CALCULUS II
NATURAL LOGARITHM EXAMPLE

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We prove that $\pi^e < e^\pi$.

We will need to use the fact that $\log x$ is an increasing function on its domain $(0, \infty)$, which is clear, since its derivative is $\frac{1}{x}$, which is positive on $(0, \infty)$.

We will also need to note that $e < \pi$.

Consider the function

$$f(x) = \frac{\log x}{x}.$$

Its derivative is

$$f'(x) = \frac{\frac{1}{x}(x) - (1)\log x}{\log^2 x} = \frac{1 - \log x}{\log^2 x}.$$

Setting this to zero, we find that $\log x = 1$, so $x = e$. The function has a local maximum there.

Now $= \log^2 x$ is positive on its domain, and for $x > e$, $\log x > 1$, so $f'(x) < 0$. Thus $f(x)$ is decreasing on the interval $[e, \infty)$, and in particular, if $x > e$, then $f(x) < f(e)$.

Since $e < \pi$, we have $f(\pi) < f(e)$. That is, $\frac{\log \pi}{\pi} > \frac{\log e}{e}$ Therefore $e \log \pi < \pi \log e$, so $\log \pi^e < \log e^\pi$. Since log is increasing, this implies that $\pi^e < e^\pi$.

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